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Early Viscous Universe with Variable Cosmological and Gravitational Constants in Higher Dimensional Space Time

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Abstract In this paper we present a number of classes of solutions of the Einstein's field equations with variable G, Λ and bulk viscosity for a Kaluza-Klein type of cosmological model. The solutions are obtained by using γ-law equation of state $p = (\gamma - 1)\rho$, where adiabatic parameter γ-varies continuously as the universe expand. A unified description of the early evolution of the universe is discussed with number of possible assumption on the bulk viscous term and gravitational constant in which an inflationary phase followed by radiation-dominated phase. We also investigate the cosmological model with constant and time dependent bulk viscosity along with constant and time dependent gravitational constant. In all cases, the cosmological constant Λ found to be positive and decreasing function of time which supports the results obtained from recent supernovae Ia observations. The important physical behaviour of the early cosmological model has also been discussed in the frame work of higher dimensional space-time.

Keywords Early universe · Gravitational and cosmological constants

1 Introduction

Dirac [1] proposed for the first time the idea of variable G on the certain physical grounds. Considerable cosmological studies on the early era have been carried out with variable G [2–9]. On the other hand, present-day astronomical observations indicate [10] that the cosmological constant Λ is extremely negligible being $\leq 10^{-56}$ cm^2. But the value of this constant should be 10^{50} times larger according to the Glashow-Weinberg-Salam model [11] for

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electro-weak unification and 10^{107} times larger according to GUT [12] for grand unification. These figures might induce one to infer that Λ was perhaps very high in the early universe when electro-weak and grand unification might have occurred, and has been steadily diminishing with the passage of time and has become extremely small (practically zero) in the present era. As might be expected, in the mean time, a number of authors [13–15] have worked out cosmologies based on time-dependent Λ in the early universe.

Recently, Carmeli and Kuzmenko [16] have shown that the cosmological relativistic theory predicts the value for cosmological constant $\Lambda = 1.934 \times 10^{-35}\ \text{s}^{-2}$. This value of Λ is excellent in agreement with the measurements recently obtained by the High-z Supernovae Team and Supernovae Cosmology Project. The main conclusion of these observations is that the expansion of the universe is accelerating. Motivated by dimensional grounds with quantum cosmology, Chen and Wu [13] have considered the variation of cosmological term as $\Lambda \propto R^{-2}$. However, a number of authors have argued in favour of the dependence $\Lambda \propto t^{-2}$. Berman [14, 17] has discussed the possibility of $\Lambda \propto t^{-2}$ by adding an additional term to the usual energy-momentum tensor, resulting in a variable Λ-*term*. In an attempt to modify the general theory of relativity, Al-Rawaf and Taha [18], Al-Rawaf [19] and Overduin and Cooperstock [20] have proposed a model with $\Lambda = \beta(\frac{\ddot{R}}{R})$, where β is constant. One of the motivations for introducing the Λ-term is to reconcile the age parameter and the density parameter of the universe with recent observational data.

Cosmological models with time-dependent G, and Λ in the solutions $\Lambda \sim R^{-2} \sim t^{-2}$, were first obtained by Bertolami [21, 22] and then number of authors studied in a series of works [23–32].

On the other hand, cosmological models of a fluid with viscosity play a significant role in the study of evolution of universe. It is well known that at an early stage of the universe when neutrino decoupling occurs, the matter behaves like a viscous fluid. The coefficient of viscosity is known to decrease as the universe expands. Viscous fluid cosmological models in early universe have been widely discussed in the literatures [33–35].

Murphy [36] has studied perfect fluid cosmological models with bulk viscosity and obtained that the big-bang singularity may be avoided in the finite past. The role of viscosity in cosmology and its influence on the appearance of the initial singularity have been studied by a number of authors [37–39]. Santos et al. [40] have derived exact solutions with bulk viscosity by considering the bulk viscous coefficient as power function of mass density. Bulk viscosity associated with the grand unified theory (GUT) may lead to an inflationary cosmology [41, 42]. Bulk viscosity can provide a phenomenological description of quantum particle creation in a strong gravitational field. Beesham [43] has studied a universe consisting of a cosmological constant $\Lambda \sim t^{-2}$ and bulk viscosity. Arbab [44, 45] has discussed a viscous model with variable G and Λ claiming that energy is conserved. Ram and Singh [46, 47] have studied early universe with bulk viscosity by using variable adiabatic parameter γ of 'gamma-law' equation of state in general relativity and Brans-Dicke's theory. Singh [48, 49] and Singh et al. [50] have discussed FRW models with variable G and Λ by using variable adiabatic parameter gamma as a function of scale factor R.

The above work motivate one to consider further work in some alternatives theories of gravitation. In this paper, by considering Kaluza-Klein type cosmological model we present a number of classes of solutions to the Einstein's field equations with variable G, Λ and bulk viscosity in the context of higher dimensional space-time. We have taken the gamma-law equation of state, but with variable adiabatic parameter as a function of scale factor and have obtained many higher dimensional cosmological solutions, which may be important in describing the early universe. A unified description of the early evolution of universe is presented with number of possible assumption on the bulk viscous term and gravitational constant in which an inflationary phase is followed by radiation-dominated phase. We also investigate the cosmological model with constant and time-dependent bulk viscosity along with constant and variable gravitational constant. The effect of viscosity is shown to affect the past and future of the universe. In all cases the cosmological constant Λ is found to be positive and decreasing function of time, which supports the results obtained from the recent supernovae Ia observations. The important physical behavior of model has also been discussed in the framework of higher dimensional space-time.

2 Higher Dimensional Model and Field Equations

We consider the Kaluza-Klein type metric

$$ds^2 = dt^2 - R^2(t)(dx^2 + dy^2 + dz^2) - B^2(t)d\psi^2, \tag{1}$$

where $B(t)$ and $R(t)$ are the scale factor.

The Einstein field equations with time-dependent cosmological and gravitational constants are given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G(t)T_{\mu\nu} + \Lambda(t)g_{\mu\nu}. \tag{2}$$

The energy-momentum tensor for viscous fluid can be written as

$$T_{\mu\nu} = (\rho + \bar{p})u_\mu u_\nu - \bar{p}g_{\mu\nu}, \tag{3}$$

where ρ is the energy density of the cosmic matter and $\bar{p}$ is the effective pressure of the fluid. The effective pressure $\bar{p}$ is related to the equilibrium pressure p by Banerjee et al. [51]

$$\bar{p} = p - \zeta\theta, \tag{4}$$

where, $\theta = \mu^a_{;a}$ and $\mu^a\mu_a = 1$, ζ stands for the coefficient of bulk viscosity that determines the magnitude of viscous stress relative to expansion.

For the metric (1) with energy momentum tensor (3) along with $B(t) = R^n$, the field equations (2) yields three independent equations

$$8\pi G(t)\rho = 3(n+1)\frac{\dot{R}^2}{R^2} - \Lambda(t), \tag{5}$$

$$8\pi G(t)\bar{p} = -(n+2)\frac{\ddot{R}}{R} - (n^2+n+1)\frac{\dot{R}^2}{R^2} + \Lambda(t), \tag{6}$$

$$8\pi G(t)\bar{p} = -3\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) + \Lambda(t), \tag{7}$$

where dot denotes derivative with respective to t and $\bar{p} = p - (3+n)\zeta H$.

The above field equations can be expressed as

$$3(n+1)\frac{\ddot{R}}{R} = -8\pi G(t)\left[\rho + (n+1)[p - (3+n)\zeta H] - \frac{n\Lambda(t)}{8\pi G(t)}\right], \tag{8}$$

$$3(n+1)\frac{\dot{R}^2}{R^2} = 8\pi G(t)\left[\rho + \frac{\Lambda(t)}{8\pi G(t)}\right]. \tag{9}$$

After eliminating $\ddot{R}$ from (8) and (9) we get,

$$\left[4\rho+2(n+1)p-\frac{(n-1)\Lambda}{4\pi G}\right]H-2(n+1)(n+3)\zeta H^2=-\left(\frac{\dot{G}}{G}\rho+\dot{\rho}+\frac{\dot{\Lambda}}{8\pi G}\right). \quad (10)$$

The usual energy-momentum conservation relation, $T^{\mu\nu}_{;\nu}=0$, leads to

$$\dot{\rho}+\left[4\rho+2(n+1)p-\frac{(n-1)\Lambda}{4\pi G}\right]H=0. \quad (11)$$

Therefore, (10) yields

$$2(n+1)(n+3)\zeta H^2=\left(\frac{\dot{G}}{G}\rho+\frac{\dot{\Lambda}}{8\pi G}\right). \quad (12)$$

In order to solve the field equations (8)–(12), we assume that the pressure p and the energy density ρ are related through the 'gamma-law' equation of state

$$p=(\gamma-1)\rho, \quad (13)$$

where γ is the adiabatic parameter. In cosmology, the value of γ is taken to be constant lying between $0\le\gamma\le 2$. In this paper, our aim is to let the parameter γ vary continuously as the universe expands and study the evolution of universe as it goes through a transition from an inflationary to a radiation-dominated phase. Carvalho [52] assumed a scale factor-dependent γ of the form

$$\gamma(R)=\frac{4}{3}\frac{A(\frac{R}{R_*})^2+(\frac{a}{2})(\frac{R}{R_*})^a}{A(\frac{R}{R_*})^2+(\frac{R}{R_*})^a}, \quad (14)$$

where A is a constant and a is the free parameter related to the power of cosmic time and lies $0\le a<1$. Here, R_* is certain reference value such that if $R\ll R_*$, inflationary phase of the evolution of the universe is obtained and for $R\gg R_*$, we have a radiation-dominated phase.

Equations (8) and (9) can be written in terms of the Hubble parameter $H=\frac{\dot{R}}{R}$, to give, respectively,

$$\dot{H}+H^2=-\frac{8\pi G(t)}{3(n+1)}\left[\rho+(n+1)[p-(3+n)\zeta H]-\frac{n\Lambda(t)}{8\pi G(t)}\right], \quad (15)$$

$$H^2=\frac{8\pi G(t)}{3(n+1)}\left[\rho+\frac{\Lambda(t)}{8\pi G(t)}\right]. \quad (16)$$

Substituting (13) into (15), we get

$$\dot{H}+H^2=-\frac{8n\pi G(t)}{3(n+1)}\left(\frac{(n+1)\gamma}{n}-1\right)\rho+\frac{8(n+3)\pi G(t)\zeta(t)H}{3}+\frac{n\Lambda(t)}{3(n+1)}. \quad (17)$$

Eliminating ρ between (16) and (17), we get the first-order differential equation

$$\dot{H}+[(n+1)\gamma+(1-n)]H^2=\frac{(n+3)}{3}8\pi G(t)\zeta(t)H+\frac{\Lambda(t)\gamma}{3}. \quad (18)$$

Now (18) can be rewritten in the form

$$H' + [(n+1)\gamma + (1-n)]\frac{H}{R} = \frac{(n+3)8\pi G(t)\zeta(t)}{3R} + \frac{\gamma\Lambda(t)}{3HR}, \tag{19}$$

where a prime denotes differentiation with respect to the scale factor R.

We consider a form of Λ as

$$\Lambda = 3\beta H^2, \tag{20}$$

where $\beta =$ constant.

Using (20) into (19), we finally get

$$H' + [(n+1-\beta)\gamma + (1-n)]\frac{H}{R} = \frac{(n+3)}{3}\frac{8\pi G(t)\zeta(t)}{R}. \tag{21}$$

The coefficient of bulk viscosity is assumed to be a simple power function of the energy density [53–55]:

$$\zeta(t) = \zeta_0 \rho^{n_0} \tag{22}$$

where ζ_0 (≥ 0) and n_0 (≥ 0) are constants.

We solve (21) by taking different physical assumptions on $G(t)$ and $\zeta(t)$ in the following sections.

3 Higher Dimensional Model with Constant Coefficient of Bulk Viscosity and *G*

We assume that $G(t) = G$ and $\zeta(t)$ = constant $=\zeta_0$, $G \neq \zeta_0$. For this case (21) reduces to

$$H' + [(n+1-\beta)\gamma + (1-n)]\frac{H}{R} = \frac{\alpha_0\zeta_0}{R}, \tag{23}$$

where $\alpha_0 = \frac{(n+3)}{3}8\pi G$.

After solving (23), we get

$$HR^{(1-n)}\left[A\left(\frac{R}{R_*}\right)^2 + \left(\frac{R}{R_*}\right)^a\right]^{\frac{2}{3}(n+1-\beta)} = \alpha_0\zeta_0 \int \frac{[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(n+1-\beta)}}{R^n} dR + C_0, \tag{24}$$

where C_0 is a constant of integration.

We now solve (24) for two early phases of universe-*inflationary* and *radiation-dominated* phases, respectively.

For *inflationary phase* ($R \ll R_*$), (24) reduces to

$$H = \frac{\alpha_0\zeta_0}{b_0} + \frac{C_0}{R^{(1-n)}(\frac{R}{R_*})^{\frac{2}{3}a(n+1-\beta)}}, \tag{25}$$

where $b_0 = [\frac{2}{3}a(n+1-\beta) + (1-n)]$.

For radiation phase ($R \gg R_*$), we get

$$H = \frac{\alpha_0\zeta_0}{b_1} + \frac{C_0}{R^{(1-n)}A^{\frac{2}{3}(n+1-\beta)}(\frac{R}{R_*})^{\frac{4}{3}(n+1-\beta)}}, \tag{26}$$

where $b_1 = [\frac{4}{3}(n+1-\beta) + (1-n)]$.

If $C_0 = 0$, H has constant value in both phases and therefore we have exponential expansion. Also we get $R = R_I \exp[\frac{\alpha_0 \zeta_0 t}{b_1}]$, $\Lambda =$ constant and $\rho =$ constant, where we have chosen the constant of integration so that, at $t = 0$, $R = R_I$. At $t = -\infty$, we have $R = 0$, but the density is finite, so that one can say that there is no singularity.

When $C_0 \neq 0$, for inflationary phase from (25), we get

$$R^{b_0} = R_*^{\frac{2}{3}a(n+1-\beta)} \left[\frac{C_1 \exp(\alpha_0 \zeta_0 t) - C_0 b_0}{\alpha_0 \zeta_0} \right], \tag{27}$$

where C_1 is the constant of integration.

If we adjust the constants C_0 and C_1 such that $C_0 = C_1 = B$, where B is another constant, the above equation reduces to

$$R^{b_0} = R_*^{\frac{2}{3}a(n+1-\beta)} \left[B \left(\frac{\exp(\alpha_0 \zeta_0 t) - b_0}{\alpha_0 \zeta_0} \right) \right]. \tag{28}$$

From (28) the solutions for the other physical parameter are,

$$H = \frac{\alpha_0 \zeta_0}{b_0} \left[1 - b_0 \exp(-\alpha_0 \zeta_0 t) \right]^{-1}, \tag{29}$$

$$\Lambda = \frac{3\beta \alpha_0^2 \zeta_0^2}{b_0^2} \left[1 - b_0 \exp(-\alpha_0 \zeta_0 t) \right]^{-2}, \tag{30}$$

$$\rho = \frac{3(n+1-\beta)}{8\pi G} \frac{\alpha_0^2 \zeta_0^2}{b_0^2} \left[1 - b_0 \exp(-\alpha_0 \zeta_0 t) \right]^{-2}. \tag{31}$$

We observe that the universe starts from a non-singular state, characterized by constant and finite initial values of R, H, Λ and ρ. From (30), it is also shown that the cosmological term is positive and a decreasing function of time (i.e. present epoch) which supports the results obtained from recent type Ia supernovae observations [56–59]. The models have a non-vanishing cosmological constant and mass density as $t \to \infty$. It is well known that with the expansion of the universe, i.e. with the increase of time t, the energy density decreases and becomes too small to be ignored. The cosmological parameter decreases and in the case $\Lambda = 0$, we find $H = 0$, the process of evolution is terminated. We also get the singularity free model for $\Lambda = 0$. Thus, (28) shows that bulk viscosity alone gives rise to exponential expansion and is able to remove the initial singularity. Therefore, it is to note that the inflationary solution can be obtained with or without the cosmological parameter due to a constant coefficient of bulk viscosity in the context of higher dimensional space-time. In case where both bulk viscosity and the cosmological constant vanish, the solution is given by

$$R = R_*^{\frac{\frac{2}{3}a(n+1)}{b_0}} (b_0 B t)^{\frac{1}{b_0}},$$

which exhibits a singularity and reduces to higher dimensional perfect fluid solutions.

An interesting property with bulk viscosity should be noted. If we assume that $t_c = (\alpha_0 \zeta_0)^{-1}$, (28) can be written as

$$R = R_*^{\frac{\frac{2}{3}a(n+1-\beta)}{b_0}} \left(\frac{B}{\alpha_0 \zeta_0} \right)^{\frac{1}{b_0}} \left[\exp\left(\frac{t}{t_c} \right) - b_0 \right]^{\frac{1}{b_0}}. \tag{32}$$

On the other hand, when one is dealing with the radiation-dominated period ($R \gg R_*$), (26) gives

$$R^{b_1} = R_*^{\frac{4}{3}(n+1-\beta)} \left[\frac{B}{A^{\frac{2}{3}(n+1-\beta)}} \left(\frac{\exp(\alpha_0\zeta_0 t) - b_1}{\alpha_0\zeta_0} \right) \right]. \tag{33}$$

The other physical parameters have the following expressions:

$$H = \frac{\alpha_0\zeta_0}{b_1} \left[1 - b_1 \exp(-\alpha_0\zeta_0 t)\right]^{-1}, \tag{34}$$

$$\Lambda = \frac{3\beta\alpha_0^2\zeta_0^2}{b_1^2} \left[1 - b_1 \exp(-\alpha_0\zeta_0 t)\right]^{-2}, \tag{35}$$

$$\rho = \frac{3(n+1-\beta)}{8\pi G} \frac{\alpha_0^2\zeta_0^2}{b_1^2} \left[1 - b_1 \exp(-\alpha_0\zeta_0 t)\right]^{-2}. \tag{36}$$

In the radiation-dominated period, the physical interpretation is similar to the case of the inflationary phase. The bulk viscosity coefficient avoids the singularity. The deceleration parameter $q = -\frac{R\ddot{R}}{\dot{R}^2}$ varies from $q = [b_0^2 \exp(-\alpha_0\zeta_0 t) - 1]$ for $R \ll R_*$ to $q = [b_1^2 \exp(-\alpha_0\zeta_0 t) - 1]$ for the radiation dominated phase. We find that $q = [b_0^2 - 1]$ or $q = [b_1^2 - 1]$ as $t \to 0$ and $q = -1$ as $t \to \infty$, which shows the inflation in the evolution of universe. The above solutions have been obtained for $0 < a < 1$.

Now we study the solution in the limit $a \to 0$ and in this case (24) becomes

$$HR^{(1-n)} \left[A\left(\frac{R}{R_*}\right)^2 + 1 \right]^{\frac{2}{3}(n+1-\beta)} = \alpha_0\zeta_0 \int \frac{[A(\frac{R}{R_*})^2 + 1]^{\frac{2}{3}(n+1-\beta)}}{R^n} dR + C_0. \tag{37}$$

4 Higher Dimensional Model with Bulk Viscosity Proportional to a Power Function of Energy Density and G

Case I $\zeta = \zeta_0\rho^{n_0}$ and $G(t) = G =$ constant. By using this values of G and ζ and (16), (21) reduces to

$$H' + [(n+1-\beta)\gamma + (1-n)]\frac{H}{R} = \alpha_1\zeta_0 \frac{H^{2n_0}}{R}, \tag{38}$$

where $\alpha_1 = \frac{(n+3)}{3} 8\pi G[\frac{3(n+1-\beta)}{8\pi G}]^{n_0}$.

Solving the above differential equation, we get

$$\frac{1}{H^{2n_0-1}[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)}}$$
$$= C_2 - (2n_0-1)\alpha_1\zeta_0 \int \frac{dR}{R^{[1+(2n_0-1)(1-n)]}[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(2n_0-1)(n+1-\beta)}}, \tag{39}$$

where $n_0 \neq \frac{1}{2}$ and C_2 is an integration constant.

We solve (39) for two early phases of evolution of universe- inflationary and radiation-dominated phases. For *inflationary phase* $R \ll R_*$, the expression for the Hubble parameter is given by

$$H = \left[C_2 \left[\left(\frac{R}{R_*} \right)^{\frac{2}{3}a(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)} \right] + \frac{\alpha_1 \zeta_0}{b_0} \right]^{\frac{1}{(1-2n_0)}}. \tag{40}$$

If $H = H_*$ for $R = R_*$, a relation between constants is given by

$$H_* = \frac{1}{[C_2[R_*^{(2n_0-1)(1-n)}] + \frac{\alpha_1 \zeta_0}{b_0}]^{\frac{1}{(2n_0-1)}}}. \tag{41}$$

When $C_2 = 0$, we obtain $H = H_*$ and $R \propto \exp(H_* t)$, which represents steady-state model for all values of n_0. It is also observed that $\Lambda = const$; $\rho = const$ and $\zeta = const$. At $t = -\infty$, we have $R = 0$, but the density is finite, so that one can say that there is no singularity. We see that bulk viscosity remains constant during inflationary phase. Depending on the value of n_0, we now proceed to study the more general solution of the model for the three different cases when $C_2 \neq 0$: (a) $2n_0 > 1$, (b) $2n_0 < 1$ and (c) $2n_0 = 1$. Out of the above three cases, the case (c) is not valid physically.

Case (a) $2n_0 > 1$. In this case, we have

$$H = \frac{1}{[C_2[(\frac{R}{R_*})^{\frac{2}{3}a(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)}] + \frac{\alpha_1 \zeta_0}{b_0}]^{\frac{1}{(2n_0-1)}}}. \tag{42}$$

We observe that H approaches a finite value as $R \to 0$, which shows the steady-state characteristic of the model. For other values of n_0 in the range $2n_0 > 1$, we get similar characteristics.

Case (b) $2n_0 < 1$. In this case, relation (40) can be written as

$$H = \left[C_2 \left[\left(\frac{R}{R_*} \right)^{-\frac{2}{3}a(1-2n_0)(n+1-\beta)} R^{-(1-2n_0)(1-n)} \right] + \frac{\alpha_1 \zeta_0}{b_0} \right]^{\frac{1}{(1-2n_0)}}. \tag{43}$$

Here, $C_2 = 0$ gives the constant value of $\frac{\dot{R}}{R}$. The model with $C_2 > 0$ explodes from $R = 0$, where $\dot{R}$ and the energy density both are infinitely large. The Hubble parameter approaches H_* asymptotically for large R. Then the universe evolves into a viscosity-dominated steady-state era. If the coefficient of bulk viscosity decays sufficiently slowly, the late epochs of the universe will be viscosity dominated, and the universe will enter a final inflationary era with steady-state character.

Case (c) For the case $2n_0 = 1$, not valid physically here, but we considered this case later in Case II.

If the coefficient of the bulk viscosity decays sufficiently slowly, the universe eventually enters a viscosity-dominated epoch at late times with constant density, i.e. if $\frac{\alpha_1 \zeta_0}{b_0} \gg C_2 (\frac{R}{R_*})^{\frac{2}{3}a(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)}$, there is exponential expansion. The solution in terms of scale factor can be obtained by integrating (40) for $H=\frac{\dot{R}}{R}$ to give

$$R=\exp\left[\left(\frac{\alpha_1\zeta_0}{b_0}\right)^{\frac{1}{1-2n_0}}t\right]=\exp(H_*t),\qquad(\zeta_0\neq 0).\tag{44}$$

On the other hand, when $C_2[(\frac{R}{R_*})^{\frac{2}{3}a(2n_0-1)(n+1-\beta)}R^{(2n_0-1)(1-n)}]\gg\frac{\alpha_1\zeta_0}{b_0}$, we get

$$R=R_*^{\frac{\frac{2}{3}a(n+1-\beta)}{b_0}}\left[\frac{b_0}{C_2}t\right]^{\frac{1}{b_0}}.\tag{45}$$

The solutions with $C_2>0$ are of most physical interest. The scale factor has the form $R\propto t^{\frac{1}{b_0}}$, which shows that the power-law expansion and the effects of viscosity are negligible. For these models, the other physical parameters in terms of scale factor have the following expressions:

$$\begin{aligned}\rho&=\frac{3(n+1-\beta)}{8\pi G}H^2\\&=\frac{3(n+1-\beta)}{8\pi G}\left[C_2\left[\left(\frac{R}{R_*}\right)^{\frac{2}{3}a(2n_0-1)(n+1-\beta)}R^{(2n_0-1)(1-n)}\right]+\frac{\alpha_1\zeta_0}{b_0}\right]^{\frac{2}{(1-2n_0)}},\end{aligned}\tag{46}$$

$$\zeta=\zeta_0\left[\frac{3(n+1-\beta)}{8\pi G}\right]^{n_0}\left[C_2\left[\left(\frac{R}{R_*}\right)^{\frac{2}{3}a(2n_0-1)(n+1-\beta)}R^{(2n_0-1)(1-n)}\right]+\frac{\alpha_1\zeta_0}{b_0}\right]^{\frac{2n_0}{(1-2n_0)}},\tag{47}$$

$$\Lambda=3\beta\left[C_2\left[\left(\frac{R}{R_*}\right)^{\frac{2}{3}a(2n_0-1)(n+1-\beta)}R^{(2n_0-1)(1-n)}\right]+\frac{\alpha_1\zeta_0}{b_0}\right]^{\frac{2}{(1-2n_0)}}.\tag{48}$$

For the radiation-dominated phase ($R\gg R_*$), (39) gives

$$H=\left[C_2\left[A\left(\frac{R}{R_*}\right)^2\right]^{\frac{2}{3}(2n_0-1)(n+1-\beta)}R^{(2n_0-1)(1-n)}+\frac{\alpha_1\zeta_0}{b_1}\right]^{\frac{1}{(1-2n_0)}}.\tag{49}$$

If $H=H_*$ for $R=R_*$, a relation between constants is given by

$$H_*=\frac{1}{[C_2A^{\frac{2}{3}(2n_0-1)(n+1-\beta)}R_*^{(2n_0-1)(1-n)}+\frac{\alpha_1\zeta_0}{b_1}]^{\frac{1}{(1-2n_0)}}}.\tag{50}$$

For $C_2=0$, we have $\frac{\dot{R}}{R}=$ constant, which represents a steady state with $(R-t)$ curve as an exponential one.

In the radiation phase, if the bulk viscosity decays sufficiently slowly, the universe eventually enters a viscosity-dominated epoch at late times with constant density, i.e., if $\frac{\alpha_1\zeta_0}{b_1} \gg C_2[A(\frac{R}{R_*})^2]^{\frac{2}{3}(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)}$, there is exponentially expansion. The solution in terms of scale factor can be obtained by integrating (49) for $H = \frac{\dot{R}}{R}$ to give

$$R = \exp\left[\left(\frac{\alpha_1\zeta_0}{b_1}\right)^{\frac{1}{1-2n_0}} t\right] = \exp(H_* t), \quad (\zeta \neq 0). \tag{51}$$

The Hubble parameter is constant and as $t \to -\infty$, $R \to 0$. The universe is infinitely old. There is no physical singularity, since the energy density assumes a finite value as $R \to 0$.

On the other hand, when $C_2[A(\frac{R}{R_*})^2]^{\frac{2}{3}(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)} \gg \frac{\alpha_1\zeta_0}{b_1}$, we get

$$R = R_*^{\frac{\frac{4}{3}(n+1-\beta)}{b_1}} \left[\frac{b_1}{C_2 A^{\frac{2}{3}(n+1-\beta)}} t\right]^{\frac{1}{b_1}}, \tag{52}$$

which shows power-law expansion of universe and the effects of viscosity are negligible. The solutions with $C_2 > 0$ are of most physical interest. One can observe that $C_2 > 0$ leads to an expansion and $C_2 < 0$ to a contraction. According, the universe is expanding or contracting one. The other physical parameters in terms of scale factor are given by

$$\rho = \frac{3(n+1-\beta)}{8\pi G}\left[C_2\left[A\left(\frac{R}{R_*}\right)^2\right]^{\frac{2}{3}(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)} + \frac{\alpha_1\zeta_0}{b_1}\right]^{\frac{2}{(1-2n_0)}}, \tag{53}$$

$$\zeta = \zeta_0\left(\frac{3(n+1-\beta)}{8\pi G}\right)^{n_0} \times \left[C_2\left[A\left(\frac{R}{R_*}\right)^2\right]^{\frac{2}{3}(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)} + \frac{\alpha_1\zeta_0}{b_1}\right]^{\frac{2n_0}{(1-2n_0)}}, \tag{54}$$

$$\Lambda = 3\beta\left[C_2\left[A\left(\frac{R}{R_*}\right)^2\right]^{\frac{2}{3}(2n_0-1)(n+1-\beta)} R^{(2n_0-1)(1-n)} + \frac{\alpha_1\zeta_0}{b_1}\right]^{\frac{2}{(1-2n_0)}}. \tag{55}$$

We observe that ρ and Λ are a decreasing function of time. For $C_2 > 0$, the viscosity coefficient is decreasing to zero for increasing R.

Case II $\zeta = \zeta_0\rho^{\frac{1}{2}}$ and $G(t) = G$. The solutions obtained in Case I are not valid for $n_0 = \frac{1}{2}$. In the case $n_0 = \frac{1}{2}$, using (16) and (20) into (21) we get,

$$H' + [(n+1-\beta)\gamma + (1-n) - \alpha_2]\frac{H}{R} = 0, \tag{56}$$

where $\alpha_2 = [\frac{(n+3)^2}{3} 8\pi G\zeta_0(n+1-\beta)]^{\frac{1}{2}}$.

Integrating (56) we get

$$H = \frac{C_3 R^{\alpha_3}}{[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(n+1-\beta)}}, \tag{57}$$

where C_3 is an integration constant and $\alpha_3 = \alpha_2 - (1-n)$.

If $H = H_*$ for $R = R_*$, a relation between constants is given by,

$$H_* = R_*^{\alpha_3} \frac{C_3}{[1+A]^{\frac{2}{3}(n+1-\beta)}}. \tag{58}$$

After integrating (57) in terms of scale factor can be written as,

$$\int \frac{[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(n+1-\beta)}}{R^{\alpha_3+1}} dR = C_3 t. \tag{59}$$

The constant of integration has been taken as zero for simplicity. For the inflationary- and radiation-dominated phases the solution of (59) respectively, given by:
for inflationary phase

$$R^{b_2} = \left[R_*^{\frac{2}{3}a(n+1-\beta)} b_2 C_3 t\right], \tag{60}$$

and for radiation phase

$$R^{b_3} = \left[\left(\frac{R_*^2}{A}\right)^{\frac{2}{3}a(n+1-\beta)} b_3 C_3 t\right], \tag{61}$$

where, $b_2 = [\frac{2}{3}a(n+1-\beta) - \alpha_3]$ and $b_3 = [\frac{4}{3}(n+1-\beta) - \alpha_3]$.

The solutions show the power-law expansion of the model. Their coefficients of viscosity have the following expressions:
for inflationary phase

$$\zeta = \zeta_0 \left(\frac{3(n+1-\beta)}{8\pi G}\right)^{\frac{1}{2}} \frac{1}{b_2} t^{-1}, \tag{62}$$

and for radiation phase

$$\zeta = \zeta_0 \left(\frac{3(n+1-\beta)}{8\pi G}\right)^{\frac{1}{2}} \frac{1}{b_3} t^{-1}. \tag{63}$$

For the expansion of universe, we must have $b_2 > 0$ for inflationary phase and $b_3 > 0$ for radiation-dominated phase. The value of β must lie in the interval $0 \le \beta < (n+1)$. The Hubble parameter in terms of scale factor is given by

$$H = C_3 R_*^{\frac{2}{3}(n+1-\beta)} R^{-b_2}, \quad (R \ll R_*), \tag{64}$$

and

$$H = C_3 \left(\frac{R_*^2}{A}\right)^{\frac{2}{3}(n+1-\beta)} R^{-b_3}, \quad (R \gg R_*). \tag{65}$$

The density and cosmological constant of the cosmic fluid are given by

$$\rho = \frac{3(n+1-\beta)}{8\pi G} H^2 = \frac{3(n+1-\beta)}{8\pi G b_2^2} t^{-2}, \tag{66}$$

$$\Lambda = \frac{3\beta}{b_2^2} t^{-2}. \tag{67}$$

Singular solutions are found for $n_0 = \frac{1}{2}$ (i.e. $\zeta = \zeta_0 \rho^{\frac{1}{2}}$) with a power-law expansion of the scale factor. It is observed that $C_3 > 0$ leads to an expansion and $C_3 < 0$ to a contraction. The universe is monotonically expanding or contracting. For collapse, the point singularity is reached in finite time. This is the singularity where $R \to 0$, $\frac{\dot{R}}{R} \to -\infty$ and $\rho \to \infty$. The energy density and cosmological constant vary inversely as the square of the age of universe whereas the effect of bulk viscosity decreases linearly as time passes. Thus, the role of bulk viscosity is more important in early stages of the evolution of universe. The deceleration parameter varies from $q = (b_2 - 1)$ for inflationary phase to $q = (b_3 - 1)$ for radiation-dominated phase. We also observed that the deceleration parameter q is positive for $b_2 > 1$, negative for $b_2 < 1$ and $q = 0$ for $b_2 = 1$. Similarly, we may describe the physical behavior for the radiation-dominated phase.

5 Higher Dimensional Model with Bulk Viscosity Proportional to the Hubble Parameter and *G*

We consider $\zeta = \zeta_0 H$ and $G(t) = G$. These relations have already been proposed as phenomenological choice in the physical literature [60, 61].

With the above assumption into (21) and after integrating we get,

$$H = \frac{C_4 R^{\alpha_5}}{[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(n+1-\beta)}}, \tag{68}$$

where, $\alpha_5 = \alpha_4 + (n-1)$ and $\alpha_4 = (\frac{n+3}{3})8\pi G\zeta_0$ and its integration leads to the solution for inflationary and radiation phases as,
for inflationary phase

$$R^{[\frac{2}{3}a(n+1-\beta)-\alpha_5]} = \left[R_*^{\frac{2}{3}a(n+1-\beta)}\left(\frac{2}{3}a(n+1-\beta)-\alpha_5\right)C_4 t\right], \tag{69}$$

and for radiation phase

$$R^{[\frac{4}{3}(n+1-\beta)-\alpha_5]} = \left[\left(\frac{R_*^2}{A}\right)^{\frac{2}{3}(n+1-\beta)}\left(\frac{4}{3}(n+1-\beta)-\alpha_5\right)C_4 t\right]. \tag{70}$$

The solutions (69) and (70) represent power-law solutions of the model. For (16), the energy density of the universe is given by

$$8\pi G\rho = 3(n+1)H^2 - \Lambda. \tag{71}$$

Using (20) in (71), we obtain

$$8\pi G\rho = 3(n+1-\beta)H^2. \tag{72}$$

We observe that the assumption $\zeta \sim H$ is equivalent to $\zeta \sim \rho^{\frac{1}{2}}$. This explains why the solutions (60), (61) and (69), (70) have the same form. The behavior of the scale factor, Hubble parameter, cosmological parameter and energy density can be interpreted similarly as in Case II of Sect. 4. But the solutions of the bulk viscosity are given by:

for inflationary phase

$$\zeta = \frac{\zeta_0}{[\frac{2}{3}a(n+1-\beta)-\alpha_5]} t^{-1}, \tag{73}$$

and for radiation phase

$$\zeta = \frac{\zeta_0}{[\frac{4}{3}(n+1-\beta)-\alpha_5]} t^{-1}, \tag{74}$$

where, $\alpha_5 = \alpha_4 + (n-1)$ and $\alpha_4 = (\frac{n+3}{3})8\pi G\zeta_0$.

We may describe the physical behavior of the solutions is similar way as we have discussed in Case II of Sect. 4.

6 Higher Dimensional Model with G Proportional to the Hubble Parameter and Bulk Viscosity

In this case, we assume

$$G = G_0 H \quad \text{and} \quad \zeta = \text{constant} = \zeta_0. \tag{75}$$

The relation $G \sim H$ has already been proposed by Dirac [1] in his large number hypothesis, which shows that gravitational constant decreases with the age of universe. Using the above assumptions in (21), we obtain

$$H' + \left[(n+1-\beta)\gamma(R) + (1-n) - \alpha_6\right]\frac{H}{R} = 0, \tag{76}$$

where, $\alpha_6 = (\frac{n+3}{3})8\pi G_0\zeta_0$.

Integration of (76) leads to

$$H = \frac{C_5 R^{\alpha_7}}{[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(n+1-\beta)}}, \tag{77}$$

where, $\alpha_7 = \alpha_6 + (n-1)$ and $\alpha_6 = (\frac{n+3}{3})8\pi G_0\zeta_0$ and C_5 is the integration constant. The above expression is similar to (57), which shows that the solution for scale factor, cosmological constant and energy density has the similar expressions as obtained in Case II of Sect. 4 but with different constant α_7. The coefficient of viscosity has been taken as a constant. But the solutions for gravitational constant are given by

$$G = \frac{G_0}{[\frac{2}{3}a(n+1-\beta)-\alpha_7]} t^{-1}, \quad (R \ll R_*), \tag{78}$$

and

$$G = \frac{G_0}{[\frac{4}{3}(n+1-\beta)-\alpha_7]} t^{-1}, \quad (R \gg R_*). \tag{79}$$

The deceleration parameter varies from $q = [\frac{2}{3}a(n+1-\beta)-\alpha_7-1]$ for $(R \ll R_*)$ to $q = [\frac{4}{3}(n+1-\beta)-\alpha_7-1]$ for $(R \gg R_*)$. It follows that q is positive for $[\frac{2}{3}a(n+1-\beta)-\alpha_7] > 1$, negative for $[\frac{2}{3}a(n+1-\beta)-\alpha_7] < 1$ and $q = 0$ for $[\frac{2}{3}a(n+1-\beta)-\alpha_7] = 1$ or for $[\frac{4}{3}(n+1-\beta)-\alpha_7]>1$, $[\frac{4}{3}(n+1-\beta)-\alpha_7]<1$ and $[\frac{4}{3}(n+1-\beta)-\alpha_7]=1$ as the case. The Hubble factor in terms of deceleration parameter can be written as

$$H=\frac{1}{1+q}t^{-1}, \tag{80}$$

where the values of Hubble parameter are given by

$$H=\frac{1}{[\frac{2}{3}a(n+1-\beta)-\alpha_7]}t^{-1}, \quad (R\ll R_*). \tag{81}$$

$$H=\frac{1}{[\frac{4}{3}(n+1-\beta)-\alpha_7]}t^{-1}, \quad (R\gg R_*). \tag{82}$$

We observe that the rate of decrease of constant of gravitation is given by

$$\frac{\dot{G}}{G}=-\frac{1}{t}. \tag{83}$$

As $t\to 0$ energy density tends to infinity and the volume tends to zero. Thus, the model has singularity at $t=0$. The cosmological constant varies inversely as the square of the age of universe, which matches with the natural dimensions. Gravitational constant and energy density decrease with cosmic time.

Now we study the solution in the limit $a\to 0$, and in this case (78) reduces to

$$H=\frac{C_5R^{\alpha_7}}{[A(\frac{R}{R_*})^2+1]^{\frac{2}{3}(n+1-\beta)}}. \tag{84}$$

In the limit of very small value of R, i.e. for $(R\ll R_*)$, the scale factor is given by

$$R^{\alpha_7}=-\frac{1}{\alpha_7 C_5}t^{-1}. \tag{85}$$

One observes that $C_5>0$ leads to a contraction. As $t\to-\infty$, we find that $R\to 0$. The model starts from infinite past with zero proper volume. Thus, for $a=0$ the universe is infinitely old and we have inverse power-law. Again, the radiation phase is described by the solution in the limit $R\gg R_*$, that is,

$$R=\left[\left(\frac{R_*^2}{A}\right)^{\frac{2}{3}(n+1-\beta)}\left(\frac{4}{3}(n+1-\beta)-\alpha_7\right)C_5t\right]^{\frac{1}{(\frac{4}{3}(n+1-\beta)-\alpha_7)}}, \tag{86}$$

which shows the power-law expansion of the universe.

7 Higher Dimensional Model with Both Bulk Viscosity and *G* Proportional to the Hubble Parameter

In this case, we assume

$$\zeta=\zeta_0H \quad \text{and} \quad G=G_0H, \tag{87}$$

where ζ_0 and G_0 are constants and $\zeta_0 \neq G_0$. Substituting (87) into (21), we obtain

$$H' + [(n+1-\beta)\gamma + (1-n)]\frac{H}{R} = \alpha_8 \zeta_0 \frac{H^2}{R}, \tag{88}$$

where $\alpha_8 = (\frac{n+3}{3})8\pi G_0$. We observe that (88) is similar to (38) with $n_0 = 1$ or $\zeta = \zeta_0 \rho$. Solving the differential equation (88), we get

$$\frac{1}{H[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(n+1-\beta)} R^{(1-n)}} = C_6 - \alpha_8 \zeta_0 \int \frac{dR}{R^{[2-n]}[A(\frac{R}{R_*})^2 + (\frac{R}{R_*})^a]^{\frac{2}{3}(n+1-\beta)}}, \tag{89}$$

where $\alpha_8 = (\frac{n+3}{3})8\pi G_0$ and $C_6 (\geq 0)$ is a constant.

The expressions of Hubble parameter for inflationary- and radiation-dominated phases are, respectively, given by

$$H = \frac{1}{[C_6[(\frac{R}{R_*})^{\frac{2}{3}a(n+1-\beta)} R^{(1-n)}] + (\frac{\alpha_8 \zeta_0}{b_0})]}, \tag{90}$$

$$H = \frac{1}{[C_6[(\frac{R}{R_*})^{\frac{4}{3}(n+1-\beta)} R^{(1-n)}] + (\frac{\alpha_8 \zeta_0}{b_1})]}, \tag{91}$$

when $C_6 \neq 0$, the explicit integration of (90) and (91) leads to the following expressions:

$$\frac{\alpha_8 \zeta_0}{b_0} \ln R + \frac{C_6}{b_0} \left(\frac{R}{R_*}\right)^{\frac{2}{3}a(n+1-\beta)} R^{(1-n)} = t + t_0, \tag{92}$$

$$\frac{\alpha_8 \zeta_0}{b_1} \ln R + \frac{C_6}{b_1} \left(\frac{R}{R_*}\right)^{\frac{4}{3}(n+1-\beta)} R^{(1-n)} = t + t_1, \tag{93}$$

where t_0 and t_1 are integration constants. However, these constants can be taken as zero for simplicity. Study of asymptotic behavior of (92) or (93) reveals that as $t \to -\infty$, $R \to 0$ and as $t \to +\infty$, $R \to \infty$. In general, it is not possible to express R explicitly as a function of time. For sufficiently small R, the first term in (92) or (93) dominates over the second, containing the viscosity term. The energy density is a decreasing function of time whereas the cosmological constant is constant in this case. If the second term dominates we get the power-law expansion $R \propto t^{\frac{1}{b_0}}$ or $R \propto t^{\frac{1}{b_1}}$ and we observe that the effect of viscosity coefficient becomes negligible. The model has singularity and the expansion continuously slows down but never reverses. Now, $C_6 = 0$ gives the steady state solution which all models approach as $t \to -\infty$.

The gravitational parameter and coefficient of bulk viscosity for inflationary- and radiation-dominated phases in terms of scale factor are given by

$$G = G_0\left[C_6\left[\left(\frac{R}{R_*}\right)^{\frac{2}{3}a(n+1-\beta)} R^{(1-n)}\right] + \frac{\alpha_8\zeta_0}{b_0}\right]^{-1}, \tag{94}$$

$$\zeta = \zeta_0\left[C_6\left[\left(\frac{R}{R_*}\right)^{\frac{2}{3}a(n+1-\beta)} R^{(1-n)}\right] + \frac{\alpha_8\zeta_0}{b_0}\right]^{-1}, \quad (R \ll R_*) \tag{95}$$

and

$$G = G_0\left[C_6\left[\left(\frac{R}{R_*}\right)^{\frac{4}{3}(n+1-\beta)} R^{(1-n)}\right] + \frac{\alpha_8\zeta_0}{b_1}\right]^{-1}, \tag{96}$$

$$\zeta = \zeta_0\left[C_6\left[\left(\frac{R}{R_*}\right)^{\frac{4}{3}(n+1-\beta)} R^{(1-n)}\right] + \frac{\alpha_8\zeta_0}{b_1}\right]^{-1}, \quad (R \gg R_*). \tag{97}$$

It is observed that the gravitational parameter and coefficient of bulk viscosity decrease to zero as R increases. The deceleration parameter for inflationary phase is found to be

$$q = \frac{C_6(\alpha_9 - 1)(\frac{R^{\alpha_9}}{R_*^{\frac{2}{3}a(n+1-\beta)}}) - \frac{\alpha_8\zeta_0}{\alpha_9}}{C_6(\frac{R^{\alpha_9}}{R_*^{\frac{2}{3}a(n+1-\beta)}}) + \frac{\alpha_8\zeta_0}{\alpha_9}}, \tag{98}$$

where $\alpha_9 = [\frac{2}{3}a(n+1-\beta) + (1-n)]$. It may be noted that although q is a function of R, but it becomes a constant $q = (\alpha_9 - 1)$ in the absence of viscosity, where q is positive for $\alpha_9 > 1$, negative for $\alpha_9 < 1$ and $q = 0$ for $\alpha_9 = 1$. In the presence of bulk viscosity, we get $q = -1$ for $C_6 = 0$, which shows the inflation of the universe. If $C_6 > 0$ and $\alpha_9 = 1$, (98) reduces to

$$q = -\left(\frac{\frac{\alpha_8\zeta_0}{\alpha_9}}{C_6(\frac{R^{\alpha_9}}{R_*^{\frac{2}{3}a(n+1-\beta)}}) + \frac{\alpha_8\zeta_0}{\alpha_9}}\right), \tag{99}$$

which indicates that the model accelerates with $q \to 0$ at the later stage of the evolution. If $C_6 > 0$ and $\alpha_9 < 1$ then $q < 0$, which shows that the expansion is accelerated throughout the evolution. At last if $C_6 > 0$ and $\alpha_9 > 1$, it follows that q is positive for $(\frac{R^{\alpha_9}}{R_*^{\frac{2}{3}a(n+1-\beta)}}) > (\frac{\frac{\alpha_8\zeta_0}{\alpha_9}}{C_6(\alpha_9-1)})$, negative for $(\frac{R^{\alpha_9}}{R_*^{\frac{2}{3}a(n+1-\beta)}}) < (\frac{\frac{\alpha_8\zeta_0}{\alpha_9}}{C_6(\alpha_9-1)})$ and $q = 0$ for $(\frac{R^{\alpha_9}}{R_*^{\frac{2}{3}a(n+1-\beta)}}) = (\frac{\frac{\alpha_8\zeta_0}{\alpha_9}}{C_6(\alpha_9-1)})$, indicating that the expansion is accelerated in the early phase and decelerated in the later phase of the evolution.

The similar behavior of the deceleration parameter can be obtained for the radiation-dominated phase.

8 Conclusion

In this paper we have consider Kaluza-Klein type cosmological model with varying gravitational cosmological constant and the bulk viscosity coefficient ζ. We have discussed the problem by using the gamma-law equation of state, in which the adiabatic parameter γ depends on scale factor R. We obtained the solution with the assumption $\Lambda \propto H^2$. For $R \gg R_*$, it enters into a radiation dominated phase and if $R \ll R_*$, inflationary phase of evolution of the universe is obtained. The first period of evolution in each model is described by inflationary phase is followed by radiation dominated phase. All the solution exhibit the feature of viscous solution with variable G and Λ. It has been also observed that cosmological constant Λ is a decreasing function of time and they approach to small positive value as time increases, which supports the results obtained from the recent Supernovae Ia observations [2–5].

The concept of viscosity has been used in a generalized way, as an expression of different dissipative processes that give rise to terms in Einstein's field equations, which appears in the energy-momentum tensor of viscous fluid as the coefficient of bulk viscosity. In this paper, we have concentrated on models with a constant coefficient of bulk viscosity and models with bulk viscosity coefficient proportional to energy density and Hubble parameter in the context of higher dimensional space-time. Some of the sections have been devoted to a variable gravitational constant. We have analyzed the consequences of the inclusion of such a dissipative term in both inflationary- and radiation-dominated phases of evolution of universe. The evolution of the universe is qualitatively similar in both phases and we summarize the main results as follows.

The solutions of the field equations can be expressed in the exact exponential form in the case of constant coefficient of bulk viscosity and the constant gravitational constant. The universe starts from non-singular state characterized by constant and finite initial values of R, H, Λ and ρ. The bulk viscosity alone may give rise to exponential expansion and is able to remove the initial singularity in some cases.

In case where the coefficient of bulk viscosity is proportional to energy density and gravitational parameter as a constant we get the solution with $0 \le n_0 \le \frac{1}{2}$ display inflationary behavior similar to the solution in Sect. 3. But the solution with $n_0 > \frac{1}{2}$ exhibit the deflationary behavior $R \propto t^{\frac{1}{b_0}}$ from when $t \to \infty$ which is asymptotically stable, because the viscous pressure decays faster than the thermodynamic pressure.

If one take the viscosity coefficient to have the general form $\zeta = \zeta_0 \rho^{n_0}$, then we obtain the solution for the case $n_0 > \frac{1}{2}$ and $n_0 < \frac{1}{2}$ and for $n_0 = \frac{1}{2}$ we obtained singular cosmological model. Our results shows that the occurrence of viscosity driven exponential inflationary behavior depends mainly on the values of n_0.

In case of the viscosity coefficient $\zeta \propto H$ and G is constant the solutions of the bulk viscosity in both the phases is inversely proportional to time. We also observed that $\zeta \sim H$ is equivalent to $\zeta \sim \rho^{\frac{1}{2}}$ and in case, $G \propto H$ and ζ is constant, we get Hubble parameter H is inversely proportional to t for both the phases. In Sect. 7, $\zeta \propto H$ and $G \propto H$, it is observed that the gravitational parameter and bulk viscosity decreases to zero as R increases.

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